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Context Independence as a Statistical Property of Hidden Variable Theories

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The context dependence of Bell local hidden variable theory is reconsidered both in its mathematical and physical justification. The compatibility of the context dependence of *individual* measurement results with the context independence of the *statistics* of measurement results is shown to warrant the consistency of the Bell framework with respect to the Gleason no-hidden-variables theorem. Finally, a sharp distinction between context dependence and (any form of) *nonlocal* dependence is defended on the background of some recent algebraic proofs of nonlocality.

1. BELL LOCAL CAUSAL THEORIES AND CONTEXTUALISM

In his fundamental paper, "On the problem of hidden variables in quantum mechanics," John Bell (1966) proved the nonexistence of dispersion-free states in quantum mechanics to follow from less questionable premises with respect to von Neumann's no-hidden-variable theorem. Two proofs were actually provided, one relying and one not relying on Gleason's theorem. However, Bell also argued that Gleason's theorem actually rules out only *noncontextual* hidden variable theories—according to which the result of a measurement of an observable depends only on the state of the system, and not also on the set of (compatible) observables that are measured with it—and it is as irrelevant as von Neumann's theorem with respect to *contextual* hidden variable theories (as is the case with Bohm's causal interpretation of quantum mechanics).

However, in his later formulation of local causal theory, suitable for the derivation of his celebrated inequality, Bell did not *explicitly* address the problem of contextualism for *his own* local hidden variable formulation, and

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contextualism is not stated as an independent condition. The question is not trivial. Had the formulation contained in Bell (1964) been noncontextual *by construction*, then the statistical incompatibility between hidden-variable-theory and quantum mechanical predictions could not even be formulated, since in this case the premises of Bell's theorem would have been inconsistent with the axioms of ordinary quantum mechanics *independent of any locality requirement.*

In the now usual formulation of the EPR-Bohm experiment, a pair $S_1 + S_2$ of spin-1/2 particles S_1 and S_2 are prepared in the singlet state and fly off in opposite directions. Spin observables σ_1 and σ_2 are measured along prefixed directions on the two particles in spacelike-separated regions. In the Bell (1964) local causal theory, where λ denotes the complete specification of the source state and $\hat{a}$ and $\hat{b}$ are unit vectors associated to two possible directions, "the result A of measuring $\sigma_1 \cdot \hat{a}$ is then determined by $\hat{a}$ and λ, and the result B of measuring $\sigma_2 \cdot \hat{b}$ in the same instance is determined by $\hat{b}$ and λ, and $A(\hat{a}, \lambda) = \pm 1$, $B(\hat{b}, \lambda) = \pm 1$" (Bell, 1964). The contributions of $\hat{a}$ and $\hat{b}$ characterize respectively the value assignments A and B as *context dependent* in general. But how is context dependence translated into a suitable Hilbert space framework for Bell theory? (Gudder, 1970; Beltrametti and Cassinelli, 1981).

All observables of S_1 and S_2 are represented by operators defined on the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Since for any pair of directions $\hat{a}, \hat{b}$ the spin operators $\hat{\sigma}_{1,\hat{a}}$ and $\hat{\sigma}_{2,\hat{b}}$ have simple spectrum, each forms a complete set of self-adjoint commuting operators. Then if $\hat{I}_1$ and $\hat{I}_2$ denote the identity operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $\{\hat{\sigma}_{1,\hat{a}} \otimes \hat{I}_2, \hat{I}_1 \otimes \hat{\sigma}_{2,\hat{b}}\}$ itself is a complete set of compatible observables of $S_1 + S_2$. This complete set determines a maximal Boolean σ-algebra $\mathbf{B}$ in the set of projectors $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, since the von Neumann algebra generated by $\{\hat{\sigma}_{1,\hat{a}} \otimes \hat{I}_2, \hat{I}_1 \otimes \hat{\sigma}_{2,\hat{b}}\}$ is maximal Abelian. This maximal Boolean σ-algebra is generated by projectors contained in the spectral decompositions of $\hat{\sigma}_{1,\hat{a}} \otimes \hat{I}_2$ and $\hat{I}_1 \otimes \hat{\sigma}_{2,\hat{b}}$.

Now it is convenient to regard $\{\hat{\sigma}_{1,\hat{a}}$ and $\hat{\sigma}_{2,\hat{b}}\}$ as fixing a certain context, and the elements of the algebra $\mathbf{B}$ as *contextual* properties of the system. Given a $\phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, representing a pure state s of $S_1 + S_2$, it is possible to define a state (probability measure) w_ϕ on $\mathbf{B}$ that is dispersion-free just on $\mathbf{B}$ (and not on any other maximal Boolean algebra induced by a different context), namely it assigns to each projector of $\mathbf{B}$ either 0 or 1. By w_ϕ one assigns a definite value to all observables represented by operators whose spectral decompositions belong to $\mathbf{B}$, and in particular to the observables represented by operators of the form $\hat{\sigma}_{1,\hat{a}} \otimes \hat{I}_2$, $\hat{I}_1 \otimes \hat{\sigma}_{2,\hat{b}}$, and $\hat{\sigma}_{1,\hat{a}} \otimes \hat{\sigma}_{2,\hat{b}}$ (the latter represents what is sometimes called a *correlation observable*). The measure w_ϕ can be considered as a mathematical representative of a *complete state* $\langle s, \lambda\rangle$, where s is the given pure quantum state of $S_1 + S_2$ and λ is supposed to represent the 'hidden variable.' The introduction of measures w_ϕ made possible by the construction of the Boolean algebra **B** amounts to the formulation of a hidden variable theory T_{hv} for the system $S_1 + S_2$: hypothetical complete states are introduced, represented by dispersion-free probability measures on **B**, in which all relevant observables of the composite system $S_1 + S_2$ turn out to have definite values. However, it is to be stressed that *such complete states are context dependent.* For, given any spin observable $\hat{\sigma}_{1,\hat{a}}$, the set

$$\{\{\hat{\sigma}_{1,\hat{a}}, \hat{\sigma}_{2,\hat{b}}\}, \{\hat{\sigma}_{1,\hat{a}}, \hat{\sigma}_{2,\hat{b}'}\}, \{\hat{\sigma}_{1,\hat{a}}, \hat{\sigma}_{2,\hat{b}''}\}, \ldots\}$$

represents a set $\{C, C', C'', \ldots\}$ of *different contexts* for possible measurements of $\hat{\sigma}_{1,\hat{a}}$ in given states. Let us denote by $[\hat{\sigma}_{1,\hat{a}}]_{\langle s,\lambda,C\rangle}$ the definite value of the observable $\hat{\sigma}_{1,\hat{a}}$ in the complete state $\langle s, \lambda\rangle$ for the context C. In our general T_{hv} for $S_1 + S_2$, $[\hat{\sigma}_{1,\hat{a}}]_{\langle s,\lambda,C\rangle} \neq [\hat{\sigma}_{1,\hat{a}}]_{\langle s,\lambda,C'\rangle}$, with $\hat{\sigma}_{2,\hat{b}}$ and $\hat{\sigma}_{2,\hat{b}'}$ incompatible. In this case, the value of $\hat{\sigma}_{1,\hat{a}}$ in $\langle s, \lambda\rangle$ need not match across the different contexts C and C'. This feature expresses the context dependence of complete states in this framework. The premises of the Bell (1964) theorem turn out to be consistent with Gleason's theorem, i.e., with the ordinary axiomatization of quantum mechanics, since the assignment of definite values to observables of the system in given states, represented by the probability measure w_ϕ, turns out to be contextual.

2. NO-GO THEOREMS AND STATISTICAL NONCONTEXTUALISM

Two classes of no-hidden-variables theorems are usually distinguished, the class of Kochen and Specker (KS) theorems and the class of Bell theorems. While KS theorems rule out *noncontextual* hidden variables, Bell theorems rule out *contextual local* hidden variables. The previous characterization of contextualism can be used to show how one of the most recent and simple no-hidden-variables theorem, due to Peres, is a KS theorem, since it rules out just *noncontextual* value assignments to observables in entangled states (Laudisa, 1997). Peres' theorem is formulated in the usual EPR-Bohm framework of a system of two spin-1/2 particles 1 + 2 for the singlet state Ψ in a four-dimensional Hilbert space (Peres, 1990).

Mermin has provided two generalizations of Peres' theorem. Although the first (let us call it M1), involving a greater number of observables, refers to the same four-dimensional Hilbert space as Peres' proof, while the second (let us call it M2) holds for observables on a eight-dimensional Hilbert space (Mermin, 1993), neither M1 nor M2 relies on the particular properties of a specific state such as the singlet state. Both M1 and M2 fall within the category of KS theorems, and M1 admittedly shares with Peres' theorem the noncontextualism assumption. However, Mermin invokes a philosophical argument of a *nonconspiratorial* sort in order to argue that M2 avoids the objection to the noncontextualism assumption contained in Peres' theorem and in M1 (Mermin, 1993, p. 810). Quantum mechanical predictions are context *in*dependent. An observable A can be measured either with mutually commuting observables $B, C, \ldots$ or with mutually commuting observables $L, M, \ldots$, while some of the $L, M, \ldots$ do not commute with all $B, C, \ldots$. Nevertheless, joint distributions for the results of an $\{A, B, C, \ldots\}$ measurement and the results of an $\{A, L, M, \ldots\}$ measurement yield the same marginal distribution for the result of the measurement of A. Now, since any viable hidden variable theory is supposed to agree on quantum mechanical predictions, Mermin contends that a *contextual* hidden variables theory would be unable to account for the reason nature should conspire to render context dependence ineffective at the level of distributions over supposedly contextual complete states (Mermin, 1993, pp. 811–12).

However, if one looks at the way in which Bell *contextual* local hidden variable theory represents quantum mechanical observables, there is no need to resort to conspiracy in nature. In Bell theory, spin observables are represented as one-point real-valued functions over the set of complete states. Now, if $\{C_1, C_2, C_3, \ldots\}$ is a set of different contexts for any EPR-Bohm observable $\sigma_{1,\hat{a}}$, we have seen that in general, for $i \neq j$, we have $[\sigma_{1,\hat{a}}]_{\langle\Psi,\lambda,C_i\rangle} \neq [\sigma_{1,\hat{a}}]_{\langle\Psi,\lambda,C_j\rangle}$. As we said, the value of $\sigma_{1,\hat{a}}$ need not match across different contexts. However, the Bell local causal theory represents $\sigma_{1,\hat{a}}$ as a one-point real-valued function $A_{1,\hat{a}}(\xi)$ on a space Ξ of complete states ξ *regardless of the contribution of other observables* $\sigma_{2,?}$ *to the definition of the different contexts.* Thus, assuming a *context-independent* distribution of complete states at the source is reasonable, since the theory prescribes no emergence of context sensitivity at the statistical level [for a similar compatibility in Bohm's causal interpretation of quantum mechanics, see Home (1994)].

3. CONTEXT DEPENDENCE AND NONLOCAL DEPENDENCE

Pagonis *et al.* (1991) have shown that a contextual hidden variable theory avoids the conclusions of M2. Furthermore, the mathematical expression of contextualism given above makes it clear how to distinguish contextualism from nonlocality. Assuming a rather general principle of locality (sometimes referred to as *Einstein locality*), no influence—appealing to which we can affect a possessed value of an observable for one system by operating on another system—can occur when the regions of the two systems are spacelike-separated. With particular reference to the notion of contextualism, two notions of locality have been further singled out: *ontological* locality and *environmental* locality (Redhead, 1987). In an EPR framework, a possessed value of an observable for one subsystem cannot depend on the features of its ontological or environmental context, specified jointly with the second subsystem: for a (nonmaximal) observable A, an ontological context is specified by the (maximal) observable B of which A is a function, whereas B specifies an environmental context whenever the measuring apparatus itself is set to measure just B. Nonlocality then entails in one way or another a dependence of observables' possessed values for one system on particular choices of observables to measure on another system that is spacelike-separated from the first. However, our characterization of contextualism need not imply that *particular* values of $\sigma_{1,\hat{a}}$ in complete states (s, λ) *depend* on whether the paired observable is $\sigma_{2,\hat{b}}$ or $\sigma_{2,\hat{b}}{}'$, even if the regions in which spin measurements are performed are spacelike-separated.

The fact that values of observables in complete states need not match in different contexts is too weak to imply nonlocality *by itself.* The coincidence of such values—notwithstanding the different contexts—is a possibility that is not excluded *a priori*, although it is not assumed in general, whereas nonlocality seems to entail an *effective* dependence of the possession of values by physical systems on operations performed is spacelike-separated regions. Contextualism actually turns out to be a *prerequisite* to test consistently local realism against quantum mechanics, if—given Gleason's theorem—we do not introduce more exotic assumptions [as in the completely nonstandard approach of Pitowsky (1989) to the contextualism issue].

Still, Mermin (1993) argues that his M2 can be *converted* into a nonlocality theorem (let us call it M3) simply by justifying noncontextualism on the basis of locality. In this case, observables like $\sigma_{1,\hat{x}}$, $\sigma_{2,\hat{y}}$, $\sigma_{3,\hat{z}}$, are defined as the spin component observables of three different spin-1/2 particles in the singlet state, spacelike-separated from one another. As above, quantum mechanics allows us to measure an observable A either with B, C, . . . or with L, M, . . ., fail to commute with all B,C,. . . . The assumption by which we take the measurement result of A to be independent of which of the $\{A, B, C, \ldots\}$ measurement or $\{A, L, M, \ldots\}$ measurement A is extracted from (i.e., noncontextualism) can be—according to Mermin—motivated by a general principle of locality if we assume that the replacement of the experimental arrangement suitable to the measurement of B, C, . . . with the experimental arrangement suitable to the measurement of L, M, . . . is performed *at a distance* from the experimental arrangement suitable to the measurement of A (Mermin, 1993, p. 812). The 'conversion' of M2 into a nonlocality theorem is thought to relate simply and effectively the class of KS theorems and the class of Bell theorems for special composite systems: in this case hidden variables would be constrained by KS 'nonlocal contextualism' to the same extent as by Bell nonlocality. However, the status of M3 as a 'nonlocality version' of M2 cannot be defended by this justification, which seems at variance with the constraints imposed by a Hilbert space description of quantum systems. For if we want to remain consistent with such a description, any nonlocality theorem must *presuppose* a contextual value assignment. Any form of 'local noncontextualism' in this framework faces inconsistency, since any noncontextual value assignment to observables in an at least three-dimensional Hilbert space is ruled out by Gleason's theorem. The more general and well-founded justification of M3 as a nonlocality theorem that complies with the Hilbert space constraints is provided by Stocks and Redhead (1996).

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